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A Dual Basis for L-Splines and Applications

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1. INTRODUCTION

Let

$$L = D^m + \sum_{i=0}^{m-1} a_i(x) D^i \tag{1.1}$$

be a linear differential operator defined on the interval $[a, b]$ with null space $N(L)$. Given a partition $\Delta = \{a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b\}$ of $[a, b]$ and a vector $\mathscr{M} = (m_1, ..., m_k)$ of integers with $1 \leqslant m_i \leqslant m$, $i = 1, 2, ..., k$, we define

$$\mathscr{S}(L; \mathscr{M}; \Delta) = \{s: s\,|_{(x_i, x_{i+1})} \in N_L,\ i = 0, 1, ..., k, \text{ and}$$

$$D_-^{j-1}s(x_i) = D_+^{j-1}s(x_i),\ j = 1, 2, ..., m - m_i,\ i = 1, ..., k\}. \tag{1.2}$$

We call $\mathscr{S}(L; \mathscr{M}; \Delta)$ the space of *L-splines.*

Although L-splines have been studied in a number of papers, compared with polynomial splines (see, e.g., the survey article of de Boor [1]) there remain a number of important gaps in their constructive theory. The purpose of this paper is to fill some of these gaps. To accomplish this, we construct a basis for $\mathscr{S}$ consisting of local support L-splines, and a corresponding dual basis. We then use the latter to study the condition number of the LB-splines,

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to study certain quasi-interpolants, and to obtain error bounds for L-spline interpolation.

2. LB-SPLINES

In this section we construct local support L-splines which are the analogs of the classical polynomial B-splines. Although there are techniques for constructing such local support splines in general (cf., Jerome and Schumakes [10]), for our purposes we need an explicit construction (which, in fact, closely resembles the construction of the polynomial B-splines).

We begin by defining an *extended partition* $\tilde{\Delta} = \{y_i\}_1^{2m+K}$, $K = \sum_{i=1}^{k} m_i$, associated with $\mathscr{S}(L; \mathscr{M}; \Delta)$:

$$y_1 = y_2 = \cdots = y_m = a, \qquad b = y_{m+K+1} = \cdots = y_{2m+K} \tag{2.1}$$

and

$$y_{m+1} \leqslant \cdots \leqslant y_{m+K} = \overbrace{x_1, ..., x_1}^{m_1}, ..., \overbrace{x_k, ..., x_k}^{m_k}. \tag{2.2}$$

Our aim is to construct a sequence $\{B_i\}_1^{m+K}$ of splines in $\mathscr{S}(L; \mathscr{M}; \Delta)$ such that

$$B_i(x) > 0 \qquad \text{for} \quad y_i < x < y_{i+m} \tag{2.3}$$

and

$$B_i(x) = 0 \qquad \text{for} \quad a \leqslant x < y_i, \quad y_{i+m} < x \leqslant b. \tag{2.4}$$

We shall define B_i explicitly in terms of a certain Green's function and with the help of certain generalized divided differences. First we need the following fact from the theory of ordinary differential equations.

LEMMA 2.1. *Given L as in* (1.1), *there exists $\delta > 0$ such that if J is any subinterval of $I = [a, b]$ with $|J| < \delta$, then N_L has a basis $\{u_i^J\}_{i=1}^m$ which forms an Extended Complete Tchebycheπ system on J.*

See, e.g., Karlin [13].

As our construction of the LB-splines involves using ECT-systems, we shall henceforth assume that $2m\bar{\Delta} < \delta$, where $\bar{\Delta} = \max_{0 \leqslant i \leqslant k} (x_{i+1} - x_i)$. Suppose that n is such that $h = (b-a)/n < \delta/2$, and let $z_\nu = a + \nu \cdot h$, $\nu = 0, 1, ..., n$. Fix $0 \leqslant \nu \leqslant n$. By construction, the interval $J_\nu = \{z_\nu, z_{\nu+2}\}$ has length less than δ, and hence by Lemma 2.1 there exists an ECT-system $\{u_i^{J_\nu}\}_{i=1}^m$ forming a basis for N_L on J_ν. For ease of notation we drop the superscript J_ν, keeping in mind that if we work on other intervals we will have to take a new basis. By the theory of ECT-systems, we may assume that $\{u_i\}_1^m$ is given in the canonical form

$$
\begin{aligned}
u_1(x) &= w_1(x) \\
u_2(x) &= w_1(x) \int_{z_\nu}^{x} w_2(s_2)\, ds_2 \\
&\cdots \\
u_m(x) &= w_1(x) \int_{z_\nu}^{x} w_2(s_2) \int_{z_\nu}^{s_2} \cdots \int_{z_\nu}^{s_{m-1}} w_m(s_m)\, ds_m \cdots ds_2 ,
\end{aligned}
\tag{2.5}
$$

where $w_i \in C^{m-i}[J_\nu]$ and $w_i(x) > 0$, $i = 1, 2,..., m$.

Associated with the ECT-system $\{u_i\}_1^m$ we shall need the differential operators

$$L_i = D_i D_{i-1} \cdots D_0 ,$$

where

$$D_i f = D(f/w_i), \qquad i = 1, 2,..., m$$

and $D_0 f = f$. We note that on the interval J_ν the differential operator L_m is well defined and has null space $N_{L_m} = N_L = \text{span}\{u_i\}_1^m$. For later reference, we note that Green's function associated with L_m and the initial conditions $L_i f(z_\nu) = 0$, $i = 0, 1,..., m - 1$ is given by

$$
\begin{aligned}
g_m(x; y) &= w_1(x) \int_y^x w_2(s_2) \int_y^{s_2} \cdots \int_y^{s_{m+1}} w_m(s_m)\, ds_m \cdots ds_2 , \qquad & x \geq y \\
&= 0, & x < y.
\end{aligned}
\tag{2.6}
$$

There is an adjunct set of functions which will play an important role in our construction. Let

$$
\begin{aligned}
u_1^*(x) &= 1, \\
u_2^*(x) &= \int_{z_\nu}^{x} w_m(s_m)\, ds_m , \\
&\cdots \\
u_m^*(x) &= \int_{z_\nu}^{x} w_m(s_m) \int_{z_\nu}^{s_m} \cdots \int_{z_\nu}^{s_3} w_2(s_2)\, ds_2 \cdots ds_m .
\end{aligned}
$$

The set $U_m^* = \{u_i^*\}_i^m$ forms an ECT-system on J. It spans the null space of the differential operator L_m^*, where

$$L_i^* = D_i^* D_{i-1}^* \cdots D_1^*,$$

$$D_i^* f = \frac{Df}{w_{m-i+1}}, \qquad i = 1, 2,..., m,$$

The ECT-system U_m^* can be extended to an ECT-system $U_{m+1}^* = \{u_i^*\}_1^{m+1}$ simply by adding the function

$$u_{m+1}^*(x) = \int_{z_\nu}^{x} w_{m+1}(s_{m+1}) \int_{z_\nu}^{s_{m+1}} \cdots \int_{z_\nu}^{s_3} w_2(s_2)\, ds_2 \cdots ds_{m+1},$$

where w_{m+1} is any positive function on J_ν.

Given a sufficiently smooth function f, we define its *divided difference with respect to* U_{m+1}^* over the points $t_1 \leqslant \cdots \leqslant t_{m+1} \in J$ by

$$[t_1, ..., t_{m+1}]_* f = \frac{D\begin{pmatrix} t_1, ..., t_{m+1} \\ u_1^*, ..., u_m^*, f \end{pmatrix}}{D\begin{pmatrix} t_1, ..., t_{m+1} \\ u_1^*, ..., u_{m+1}^* \end{pmatrix}}, \tag{2.7}$$

where

$$D\begin{pmatrix} t_1, ..., t_{m+1} \\ u_1^*, ..., u_{m+1}^* \end{pmatrix} = \det(L_{d_j}^* u_i^*(t_j))_{i,j=1}^{m+1}$$

with

$$d_j = \max\{k: t_j = t_{j-1} = \cdots = t_{j-k}\}, \qquad j = 1, ..., m.$$

The numerator in (2.7) is defined similarly. For convenience of notation, we write

$$D_{U_{m+1}^*}(t_1, ..., t_{m+1}) = D\begin{pmatrix} t_1, ..., t_{m+1} \\ u_1^*, ..., u_{m+1}^* \end{pmatrix}.$$

This determinant is never 0 for $\{t_i\}_1^{m+1}$ in J_ν since U_{m+1}^* is an ECT-system on this interval.

We are finally ready to define the *LB*-splines B_i for all i such that $z_\nu \leqslant y_i < z_{\nu+1}$. For any such i, let

$$B_i(x) = (-1)^m \alpha_i [y_i, ..., y_{i+m}]_* g_m(x; y), \qquad i = 1, ..., m + K, \tag{2.8}$$

where

$$\alpha_i = \frac{D_{U_{m+1}^*}(y_i, ..., y_{i+m}) \cdot D_{U_{m+1}^*}(y_{i+1}, ..., y_{i+m-1})}{D_{U_m^*}(y_{i+1}, ..., y_{i+m}) \cdot D_{U_m^*}(y_i, ..., y_{i+m-1})}, \tag{2.9}$$

and $g_m(x; y)$ is Green's function defined in (2.6).

The construction outlined above can be repeated for each $\nu = 0, 1, ..., n-1$ to construct a full set of $m + K$ *LB*-splines $\{B_i\}_1^{m+K}$. The following theorem summarizes their basic properties.

THEOREM 2.2. *For each $i = 1, 2, \ldots, m + K$, the spline B_i is the unique (aside from a constant multiplier) L-spline satisfying properties (2.3)–(2.4). Moreover,*

$$\sum_{i=1}^{m+K} B_i(x) = u_1^\nu(x), \qquad \nu = 0, \tag{2.10}$$

and

$$\sum_{i=1}^{m+K} B_i(x) \leqslant u_1^\nu(x) + u_1^{\nu-1}(x), \qquad \nu = 1,\ldots, n, \tag{2.11}$$

where in general $\{u_i^\nu\}_1^m$ is the ECT-system associated with the interval $J_\nu = [z_\nu, z_{\nu+2}]$.

Proof. For each i, if J_ν is the associated interval used in the construction of B_i, then on J_ν we have $B_i \in \mathscr{S}(N_{L_m}; \mathscr{M}; \Delta)$, and it follows that $B_i \in \mathscr{S}(L; \mathscr{M}; \Delta)$. Moreover, on J_ν, B_i is in fact a Tchebycheffian B-spline (cf. [15–17]), and thus satisfies properties (2.3)–(2.4). If $\tilde{B}_i$ were another element of $\mathscr{S}(N_{L_m}; \mathscr{M}; \Delta)$ satisfying (2.3)–(2.4), then for some choice of β, $g = B_i - \beta\tilde{B}_i$ would have a zero in the interval (y_i, y_{i+m}), which by Theorem 5.1 of [17] can happen only if $g = 0$; i.e., $B_i = \beta\tilde{B}_i$.

To show (2.10)–(2.11) we rely on results on Tchebycheffian splines. Let $I_\nu = [z_\nu, z_{\nu+1}]$, $\nu = 0,\ldots, n-1$. Then with the normalization (2.8), it is known (cf. Marsden [15]) that for $x \in I_\nu$ the sum of all $B_i(x)$ with y_i in $I_{\nu-1}$ is at most $u_1^{\nu-1}(x)$. Similarly, for $x \in I_\nu$, the sum of $B_i(x)$ with $y_i \in I_\nu$ is at most $u_1^\nu(x)$. Property (2.11) follows. In I_0 we have a complete set of Tchebycheffian B-splines, and hence the sum is exactly $u_1^0(x)$ for all $x \in I_0$. ∎

Theorem 2.2 shows that each of the LB-splines $\{B_i\}_1^{m+K}$ defined in (2.8) is an element of $\mathscr{S}(L; \mathscr{M}; \Delta)$. It can be shown by a simple direct argument that $\mathscr{S}(L; \mathscr{M}; \Delta)$ is a $m + K$-dimnsional linear space (cf. [16]). Since it follows from our construction of a dual set of linear functionals $\{\lambda_i\}_1^{m+K}$ in the next section that the $\{B_i\}_1^{m+K}$ are linearly independent, we conclude that the LB-splines $\{B_i\}_1^{m+K}$ form a *basis* for $\mathscr{S}(L; \mathscr{M}; \Delta)$.

In the remainder of this section we explore the connection between the LB-splines and the classical polynomial B-splines. We begin with a lemma about determinants formed from an ECT-system.

LEMMA 2.3. *Let $U_m = \{u_i\}_1^m$ be an ECT-system as in (2.5), and let $a \leqslant t_1 \leqslant \cdots \leqslant t_m \leqslant b$. Then the determinant $D = D_{U_m}(t_1, \ldots, t_m)$ can be written as a multiple integral (over positively oriented subintervals of $[a, b]$) whose integrand involves only products of the functions $w_1, \ldots, w_m$. The same assertion holds for the determinant*

$$L_k D_{U_m}(t_1, \ldots, t_{m-1}, x) \tag{2.12}$$

for all $k = 0, 1, \ldots, m-1$ and all $a \leqslant x \leqslant b$, $x \neq t_i$, $i = 1, \ldots, m-1$.

Proof. To establish this lemma we need the concept of a reduced system associated with U_m. Following Karlin [13], we call $U_m{}^j = \{u_{j,i}\}_{i=1}^{m-j}$ the *jth reduced system associated with* U_m, where

$$u_{j,1}(x) = w_{j+1}(x),$$

$$u_{j,2}(x) = w_{j+1}(x) \int_a^x w_{j+2}(s_{j+2})\, ds_{j+2},$$

$$\cdots$$

$$u_{j,m-j}(x) = w_{j+1}(x) \int_a^x \cdots \int_a^{s_{m-1}} w_m(s_m)\, ds_m \cdots ds_{j+2}.$$

Clearly $U_m{}^j$ is an ECT-system.

We proceed by induction on m. Suppose that

$$t_1 \leqslant t_2 \leqslant \cdots \leqslant t_m = \overbrace{\tau_1, \ldots, \tau_1}^{l_1} < \cdots < \overbrace{\tau_d, \ldots, \tau_d}^{l_d}. \tag{2.13}$$

Then factoring $w_1(\tau_1)$ out of the first row of D, $w_1(\tau_2)$ out of the $l_1 + 1$st row, etc., and subtracting each row with a 1 in the first column from its predecessor with a 1 in the first column we obtain (after expanding about the first column)

$$D = w_1(\tau_1) \cdots w_1(\tau_d)$$
$$\cdot \int_{\tau_1}^{\tau_2} \cdots \int_{\tau_{d-1}}^{\tau_d} D_{U_m{}^1}(\overbrace{\tau_1, \ldots, \tau_1}^{l_1-1}, s_1, \overbrace{\tau_2, \ldots, \tau_2}^{l_2-1}, s_2, \ldots, \overbrace{\tau_d, \ldots, \tau_d}^{l_d-1})\, ds_1 \cdots ds_{d-1}. \tag{2,14}$$

Now by the inductive hypothesis the integrand is a multiple integral of products of the w's, and our first assertion has been established.

Consider now the determinant (2.12). By the definition of the L_k, we note that

$$\begin{aligned} L_k u_i(x) &= 0, & i &= 1, 2, \ldots, k, \\ &= u_{k,i-k}(x), & i &= k+1, \ldots, m. \end{aligned}$$

Thus the row corresponding to x in the determinant has zeros in the first k columns. Thus if k steps of the above process are carried out, this row is not disturbed, and we end up with a determinant formed from functions in the kth reduced system $U_m{}^k$. Now the argument used above applies to complete the proof. ∎

We can now give upper and lower bounds on the determinants discussed in Lemma 2.3. Our bounds involve the classical *Vandermande determinants* defined by

$$V(t_1, \ldots, t_m) = D_{V_m}(t_1, \ldots, t_m),$$

where $V_m = \{1, x, \ldots, x^{m-1}\}$; i.e., $w_1(x) = 1$ and $w_i(x) = x^{i-1}$, $i = 2, \ldots, m$.

LEMMA 2.4. *Let $U_m = \{u_i\}_1^m$ be an ECT-system as in (2.5), and let*

$$\underline{M}_i = \min_{a \leqslant x \leqslant b} w_i(x), \qquad \overline{M}_i = \max_{a \leqslant x \leqslant b} w_i(x), \tag{2.15}$$

$i = 1, 2, \ldots, m$. Then there is a positive function $C_1(M_1, \ldots, M_m)$ such that for all $a \leqslant t_1 < \cdots < t_m \leqslant b$,

$$\underline{C}_1 V(t_1, \ldots, t_m) \leqslant D_{U_m}(t_1, \ldots, t_m) \leqslant \overline{C}_1 V(t_1, \ldots, t_m), \tag{2.16}$$

where $\underline{C}_1 = C_1(\underline{M}_1, \ldots, \underline{M}_m)$ and $\overline{C}_1 = C_1(\overline{M}_1, \ldots, \overline{M}_m)$. Moreover, for $a \leqslant x \leqslant b$,

$$\begin{aligned} \underline{C}_1 \mid D^k V(t_1, \ldots, t_{m-1}, x)| &\leqslant | L_k D_{U_m}(t_1, \ldots, t_{m-1}, x)| \\ &\leqslant \overline{C}_1 \mid D^k V(t_1, \ldots, t_{m-1}, x)|. \end{aligned} \tag{2.17}$$

Proof. In view of Lemma 2.3, we get an upper bound on $D_{U_m}(t_1, \ldots, t_m)$ if we replace each weight function w_i by $\overline{M}_i$, $i = 1, 2, \ldots, m$. But if $w_1, \ldots, w_m$ are all constant, then the functions $u_1, \ldots, u_m$ are constant multiples of $1, x, \ldots, x^{m-1}$, and the corresponding determinant is a constant multiple of the Vandermonde. This proves the upper bound in (2.16). To get the lower bound we substitute $\underline{M}_i$ for each w_i, $i = 1, 2, \ldots, m$. The bounds in (2.17) are established in the same way. ∎

The following theorem shows that the LB-splines $\{B_i\}_1^{m+K}$ can be regarded as perturbations of the classical normalized polynomial B-splines defined by

$$N_i^m(x) = (y_{i+m} - y_i)(-1)^m [y_i, \ldots, y_{i+m}](y - x)_+^{m-1}, \qquad i = 1, \ldots, m + K,$$

where

$$[y_i, \ldots, y_{i+m}]f = \frac{D\begin{pmatrix} y_i, \ldots, y_{i+m} \\ 1, \ldots, x^{m-1}, f \end{pmatrix}}{\begin{pmatrix} y_i, \ldots, y_{i+m} \\ 1, \ldots, x^m \end{pmatrix}}. \tag{2.18}$$

is the usual divided difference.

THEOREM 2.5. *Fix* $\{y_i\}_1^{m+K}$, *and consider a sequence of differential operators as in* (1.1) *with coefficients* $a_{0,n}, \ldots, a_{m-1,n}$. *Let* $B_{1,n}(x), \ldots, B_{m+K,n}(x)$ *be the associated LB-splines. Then*

$$\| a_{i,n} \|_\infty \to 0, \qquad i = 0, \ldots, m-1 \text{ as } n \to \infty, \tag{2.19}$$

implies

$$\| B_{i,n} - N_i^m \|_\infty \to 0, \qquad i = 1, 2, \ldots, m + K.$$

Proof. From the theory of ordinary differential equations we know that (2.19) implies that the null spaces of the corresponding linear differential operators tend to $V_m = \{1, x, \ldots, x^{m-1}\}$. At the same time the w's defining the ECT-systems spanning these null spaces tend to the values $w_1 = 1$, $w_i = i - 1$, $i = 2, \ldots, m$ associated with the functions V_m. It follows that Green's function $g_m(x; y)$ in (2.6) tends to $(y - x)_+^{m-1}$, while by Lemma 2.4 the divided difference (2.7) tends to the usual divided difference (2.18). We conclude that $B_{i,n}$ converges uniformly to N_i^m for each $i = 1, 2, \ldots, m + K$. ∎

This perturbation result can be used to extend some of the properties of polynomial *B*-splines to *LB*-splines for L sufficiciently near D^m. We do not make further use of this observation, however, as the dual basis constructed in the following section is a considerably more potent tool.

3. A Dual Basis

Suppose that $2m\bar{\Delta} < \delta$, where $\bar{\Delta}$ and δ are as in Section 2, and let $\{B_i\}_1^{m+K}$ be the *LB*-splines constructed in (2.8). In this section we construct a *dual basis* for $\{B_i\}_1^{m+K}$; i.e., a set of linear functionals $\{\lambda_i\}_1^{m+K}$ such that

$$\lambda_i B_j = \delta_{ij}, \qquad i, j = 1, 2, \ldots, m + K.$$

Fix $1 \leqslant i \leqslant m + K$, and let J_ν be the subinterval of $[a, b]$ used in the construction of B_i. Let L_m^* be the operator defined in 2 associated with this interval. and let U_m^* be the associated ECT-system. Set

$$\varphi_i(x) = \frac{D_{U_m^*}(y_{i+1}, \ldots, y_{i+m-1}, x)}{D_{U_{m-1}^*}(y_{i+1}, \ldots, y_{i+m-1})}.$$

Let $T(x) \in L_\infty^m(\mathbb{R})$ be such that

$$\begin{aligned} T(x) &\equiv 0 && \text{for } x \leqslant 0, \\ T(x) &\equiv 1 && \text{for } x \geqslant 1, \\ \| D^j T \|_{L_\infty[0,1]} &\leqslant C_j < \infty, && j = 0, 1, \ldots, m-1. \end{aligned}$$

T is a kind of transition function; it can be constructed by integrating the perfect B-spline (see [3]). With

$$\psi_i(x) = \varphi_i(x) \cdot T\left(\frac{x - y_i}{y_{i+m} - y_i}\right) \tag{3.1}$$

we now define a linear functional for functions $f \in L_1[a, b]$ by

$$\lambda_i f = \int_{y_i}^{y_{i+m}} f(x)\, L_m^* \psi_i(x)\, dx. \tag{3.2}$$

THEOREM 3.1. *The sequence of linear functionals* $\{\lambda_i\}_1^{m+K}$ *forms a dual basis for* $\{B_i\}_1^{m+K}$.

Proof. By the support property (2.4) of the B-splines, it follows directly from (3.2) that

$$\lambda_i B_j = 0, \qquad j = 1, 2,..., i - m, i + m,..., m + K.$$

Now it is easily shown (cf. [16]) that if $L_m^* f \in L_1[y_j, y_{j+m}]$, then

$$[y_j, ..., y_{j+m}]_* f = \int_{y_j}^{y_{j+m}} \frac{B_j(x)\, L_m^* f(x)}{\alpha_j}\, dx. \tag{3.3}$$

This implies that

$$\lambda_i B_j = \alpha_j [y_j, ..., y_{j+m}]_* \psi_i .$$

Since $\psi_i(x)$ vanishes at $y_{i+1}, ..., y_{i+m-1}$ (with derivatives when there are repetitions), it follows that

$$\lambda_i B_j = 0, \qquad j = i + 1 - m, ..., i - 1.$$

The functions ψ_i and φ_i agree on the point set $y_j, ..., y_{j+m}$ (along with their derivatives in case of repeated y's) for $j > i$. Since $\varphi_i \in U_m^*$, this implies

$$\lambda_i B_j = [y_j, ..., y_{j+m}]_* \varphi_i = 0, \qquad j = i + 1, ..., i + m - 1.$$

It remains to check that $\lambda_i B_i = 1$. By construction $\alpha_i \psi_i$ agrees with the function

$$\xi_i(x) = \frac{D_{U_{m+1}^*}(y_i, ..., y_{i+m-1}, x)}{D_{U_m^*}(y_i, ..., y_{i+m-1})}$$

on the point set $y_i ,..., y_{i+m}$. But then

$$\lambda_i B_i = [y_i ,..., y_{i+m}]_* \xi_i = 1,$$

and the theorem is proved. ∎

The following theorem gives bounds on the linear functionals $\{\lambda_i\}_1^{m+K}$.

THEOREM 3.2. *The linear functionals* $\{\lambda_i\}_1^{m+K}$ *defined in* (3.2) *satisfy* $(\bar{\Delta}_i := (y_{i+m} - y_i))$

$$| \lambda_i f | \leqslant C\bar{\Delta}_i^{-1/p} \| f \|_{L_p[y_i, y_{i+m}]} , \tag{3.4}$$

$i = 1, 2,..., m + K$ *and any* $1 \leqslant p \leqslant \infty$, *where* C *is a constant depending only on* m *and the quantities in* (2.15).

Proof. Applying the Hölder inequality to (3.2), we have (with $I_i = [y_i , y_{i+1}]$)

$$| \lambda_i f | \leqslant \| f \|_{L_p[I_i]} \cdot \| L_m^* \psi_i \|_{L_{p'}[I_i]}$$

with $1/p + 1/p' = 1$. Using the Leibniz rule, it is easy to show by induction on m that

$$L_m^* \psi_i(x) = \sum_{k=0}^{m} \frac{c_k(x)}{d_k(x)} D^{m-k} T \left(\frac{x - y_i}{y_{i+m} - y_i} \right) L_k^* \varphi_i(x),$$

where $c_k(x)$ depends only on the values of $\{w_i\}_1^m$ and their derivatives while $d_k(x)$ depends only on powers of $\{w_i\}_1^m$. Since

$$\left\| D^{m-k} T \left(\frac{x - y_i}{y_{i+m} - y_i} \right) \right\|_{L_\infty[I_i]} \leqslant \frac{C_{m-k}}{(y_{i+m} - y_i)^{m-k}} , \qquad k = 0, 1,..., m,$$

and $L_m^* \varphi_i = 0$, this implies

$$\frac{| \lambda_i f |}{\| f \|_{L_p[I_i]}} \leqslant (y_{i+m} - y_i)^{1-1/p} \max_{0 \leqslant k \leqslant m-1} \frac{\| L_k^* \varphi_i \|_{L_\infty[I_i]}}{(y_{i+m} - y_i)^{m-k}} . \tag{3.5}$$

By Lemma 2.4, and the definition of φ_i ,

$$| L_k^* \varphi_i(x) | \leqq C_3 | D^k \Phi_i(x) |, \qquad x \in I_i ,$$

where

$$\Phi_i(x) = \frac{V(y_{i+1} ,..., y_{i+m-1} , x)}{V(y_{i+1} ,..., y_{i+m-1})}$$

Now suppose

$$y_{i+1},\dots,y_{i+m-1} = \overbrace{\tau_1,\dots,\tau_1}^{l_1} < \cdots < \overbrace{\tau_d \cdots \tau_d}^{l_d}$$

Then

$$\Phi_i(x) = \prod_{i=1}^{d} (x - \tau_i)^{l_i}$$

and it follows that

$$| D^k\Phi_i(x)| \leqslant C_4 \bar{\Delta}_i^{m-1-k}, \qquad k = 0, 1,\dots, m-1.$$

Substituting this in (3.5) yields (3.4). ∎

The dual basis $\{\lambda_i\}_1^{m+K}$ for $\{B_i\}_1^{m+K}$ can now be used to examine the conditioning of this basis.

THEOREM 3.3. *Fix $1 \leqslant p \leqslant \infty$. For $i = 1, 2,\dots, m + K$ let*

$$B_{i,p}(x) = \bar{\Delta}_i^{-1/p} B_i(x). \tag{3.6}$$

Then there exist constants $0 < C_1, C_2 < \infty$ depending only on m and the quantities $\{\underline{M}_i, \overline{M}_i\}_1^m$ in (2.15) (with respect to the finitely many intervals J_ν) such that

$$\left(\sum_{i=1}^{m+K} | c_i |^p\right)^{1/p} \leqslant C_1 \left\| \sum_{i=1}^{m+K} c_i B_{i,p} \right\|_{L_p[a,b]} \leqslant C_2 \left(\sum_{i=1}^{m+K} | c_i |^p\right)^{1/p} \tag{3.7}$$

for all sets of coefficients $c_1,\dots, c_{m+K}$.

Proof. Let $s = \sum_{i=1}^{m+K} c_i B_{i,p}$. Then

$$\sum_{i=1}^{m+K} | c_i |^p = \sum_{i=1}^{m+K} | \lambda_i s |^p \bar{\Delta}_i \leqslant C \sum_{i=1}^{m+K} \| s \|_{L_p[I_i]}^p \leqslant C \| s \|_{L_p[a,b]}^p .$$

Conversely, by our normalization of the B-splines (cf. (2.10)–(2.11))

$$| B_{i,p}(x)| \leqslant M_1 \cdot \bar{\Delta}_i^{-1/p}, \qquad i = 1, 2,\dots, m + K.$$

Thus,

$$\int_a^b | s(x)|^p \, dx = \sum_{j=0}^{K} \int_{x_j}^{x_{j+1}} \left| \sum_{i=j-m+1}^{j} c_i B_{i,p} \right|^p$$

$$\leqslant \sum_{j=0}^{K} \int_{x_j}^{x_{j+1}} \max_{j-m+1\leqslant i\leqslant j} \| B_{i,p} \|_{L_\infty[I_i]}^p \, m^{p-1} \sum_{i=j-m+1}^{j} | c_i |^p$$

$$\leqslant C \sum_{i=1}^{m+K} | c_i |^p. \quad ∎$$

4. Applications

In this section we illustrate how the dual basis constructed in the previous section can be used to derive properties of L-splines. We begin by defining a useful approximation operator. Let $\{B_i\}_1^{m+K}$ and $\{\lambda_i\}_1^{m+K}$ be as in 3. Then for every $f \in L_p[a, b]$ we define

$$Qf(x) = \sum_{i=1}^{m+K} \lambda_i f B_i(x).$$

Q is a mapping of $L_p[a, b]$ onto the spline space $\mathscr{S}(L; \mathscr{M}; \Delta)$. In analogy with similar operators which have been constructed for polynomial splines, we call it a *quasi-interpolant* (cf. [1, 4, 13]).

THEOREM 4.1. *Q is a bounded linear projector of $L_p[a, b]$ onto $\mathscr{S}(L; \mathscr{M}; \Delta)$. Moreover, if $f \in L_p^m(a, b) = \{f\colon D^j f \in L_p(a, b), 0 \leqslant j \leqslant m\}$ then for $j = 0, 1,..., m-1$,*

$$\| D^j(f - Qf)\|_q \leqslant C \frac{\bar{\Delta}^{m+1/q-1/p}}{\underline{\Delta}^j} \| Lf \|_p \tag{4.1}$$

for all $1 \leqslant p \leqslant q \leqslant \infty$. Here C is a constant which is independent of both f and Δ while $\underline{\Delta}$ is defined by $\underline{\Delta} = \min_{1 \leqslant i = m+k} (y_{i+1} - y_i)$.

Proof. Fix $0 \leqq j \leqq m-1$ and $a \leqq t \leqq b$. By the generalized Taylor expansion associated with Green's function g_m^* corresponding to the operator L_m^*, we have

$$f(x) = p_f(x) + \int_t^x g_m^*(x; y)\, Lf(y)\, dy, \tag{4.2}$$

with $p_f \in U_m$ such that

$$D^j f(t) = D^j p_f(t), \qquad j = 0, 1,..., m-1.$$

Since clearly $Qg = g$ for all $g \in \mathscr{S}(L; m; \Delta)$, Theorem 3.2 implies

$$\begin{aligned} | D^j(f - Qf)(t)| &= | D^j(p_f - Qf)(t)| \\ &= \sum_{i=1}^{m+k} \lambda_i(p_f - f)\, D^j B_i(t)| \\ &\leqq C \sum_{i=\nu+1-m}^{\nu} \bar{\Delta}_i^{-1/p} \| p_f - f \|_{L_p(I_i)} \mid D^j B_i(t)|, \end{aligned}$$

where $\nu = \nu(t)$ is such that $t \in I_\nu$, where in general $I_i = [y_i, y_{i+1})$. Now applying the Hölder inequality to (4.2), and observing that

$$| g_m^*(x; t)| \leqq C \mid x - t \mid^{m-1} \qquad \text{for all } x \text{ and } t,$$

one easily sees that

$$\bar{\Delta}_i^{-1/p} \| f - p_f \|_{L_p[I_i]} \leqq C\bar{\Delta}^{m-1/p} \| Lf \|_{L_p[I_i]} .$$

By a Bernstein–Markov-type inequality for L-splines (cf. [12]),

$$| D^j B_i(t) | \leqq \frac{C}{\underline{\Delta}^j} \| B_i \|, \qquad j = 0, 1,..., m - 1.$$

Combining these facts, we obtain

$$| D^j(f - Qf)(t) | \leqslant C \frac{\bar{\Delta}^{m-1/p}}{\underline{\Delta}^j} \sum_{i=\nu+1-m}^{\nu} \| Lf \|_{L_p[I_i]} .$$

and integrating the qth power,

$$\| D^j(f - Qf) \|_{L_q[I_i]} \leqslant C \frac{\underline{\Delta}^{m-1/p+1/q}}{\underline{\Delta}^j} \sum_{r=i+1-m}^{i} \| Lf \|_{L_p[I_r]} .$$

Summing this inequality over $i = m,..., m + K$ and using the Jensen inequality we obtain (4.1). ∎

As a second application of our dual linear functionals, we now give error bounds for best interpolating L-splines. Let $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_{2m+K}$ be a set of points with $y_i < y_{i+m}$, all i, and such that

$$y_1 = \cdots = y_m = a, \qquad b = y_{m+K+1} = \cdots = y_{2m+K} .$$

For each $i = 1, 2,..., m + K$, define

$$d_i = \max\{j: y_i = \cdots = y_{i-j}\}$$

and

$$\mu_i f = D^{d_i} f(y_i).$$

Associated with these linear functionals we may consider the *best interpolation problem*

$$\underset{u \in U_F}{\text{minimize}} \| Lu \|_{L_2[a,b]} . \tag{4.3}$$

where F is a given function in $L_2^m[a, b]$, and

$$U_F = \{ f \in L_2^m[a, b]: \mu_i f = \mu_i f, i = 1, 2,..., m + K\}. \tag{4.4}$$

Such best interpolation problems have been considered by many authors (see, e.g., [9, 11] and the references therein).

It is known (cf. [11]) that if $K \geqslant m$ then problem (4.3) has a unique solution s (called an *L-spline*) and s is uniquely characterized by the fact that it lies in U_F and satisfies the orthogonality condition

$$\int_a^b Ls(x)\, Lg(x)\, dx = 0 \qquad \text{for all } g \in U_0 .$$

The spline s can be determined numerically if we introduce the B-splines $\{B_i^*\}_1^{m+K}$ associated with the partition $\{y_i\}_1^{m+K}$ and the operator L^*. In this case, it is known that

$$Ls(x) = \sum_{i=1}^{m+K} c_i B_{i,2}^*(x), \tag{4.5}$$

To determine the coefficients $\{c_i\}$, we observe that (cf. (3.3)) for each $j = 1, 2,..., m + K$,

$$\begin{aligned}\int_a^b B_j^*(x)\, LF(x)\, dx &= [y_j ,..., y_{j+m}]_* F = [y_j ,..., y_{j+m}]_* s \\ &= \sum_{i=1}^{m+K} c_i \int_a^b B_j^*(x)\, B_{i,2}^*(x)\, dx.\end{aligned}$$

This implies that

$$Ac = r, \tag{4.6}$$

where $c = [c_1 ,..., c_{m+K}]^T$, $r = (r_1 ,..., r_{m+K})^T$, and $A = (A_{ij})_{i,j=1}^{m+K}$, with

$$\begin{aligned} r_j &= \int_a^b B_{j,2}^*(x)\, LF(x)\, dx, \\ A_{ij} &= \int_a^b B_{j,2}^*(x)\, B_{i,2}^*(x)\, dx. \end{aligned} \tag{4.7}$$

(Here we have used the normalized versions of the B-splines introduced in (3.6).)

As a first step towards establishing error bounds for $F - s$, we have the following theorem.

THEOREM 4.2. *Let* $1 \leqq p \leqq \infty$. *Then*

$$\| Ls \|_p \leqq C \| LF \|_p (\bar{\Delta}/\underline{\Delta})^{|1/p - 1/2|} \tag{4.8}$$

where C is a constant independent of F, s and Δ. (*In the following, C is always such a constant but may have a different value in each line*).

Proof. First we show that the condition number $K = \| A \|_2 \cdot \| A^{-1} \|_2$ *is bounded independently of Δ. For any vector* $\gamma \in R^{m+K}$, (3.7) (applied to the B-splines $B^*_{i,2}$) implies

$$\| A\gamma \|_2 \geqq \left\| \sum \gamma_j B^*_{j,2} \right\|_2^2 \Big/ \| \gamma \|_2 \geqq C \| \gamma \|_2 .$$

This shows $\| A^{-1} \|_2$ is bounded. On the other hand,

$$\begin{aligned} \| A\gamma \|_2 &= \sup_{\beta \neq 0} \frac{|(A\gamma, \beta)|}{\| \beta \|_2} \\ &= \sup_{\beta} \left| \int \sum \gamma_j B^*_{j,2} \sum \beta_i B^*_{i,2} \right| \Big/ \| \beta \|_2 \\ &\leqq \sup_{\beta} \left\| \sum \gamma_j B^*_{j,2} \right\|_2 \left\| \sum \beta_i B^*_{i,2} \right\|_2 \Big/ \| \beta \|_2 \\ &\leqq C \| \gamma \|_2 , \end{aligned}$$

where we have again used (3.7). This shows $\| A \|_2$ is bounded. By a theorem of Demko [6], the elements b_{ij} of the inverse matrix A^{-1} then satisfy

$$| b_{ij} | \leqq C\theta^{|i-j|}$$

for some constants C and $\theta \in [0, 1)$ which are independent of Δ. Thus, we have

$$\begin{aligned} \left(\sum_k | c_k |^p \right)^{1/p} &\leqq C \left(\sum_k \left(\sum_j \theta^{|k-j|} | r_j | \right)^p \right)^{1/p} \\ &\leqq C \sum_j \theta^{|j|} \left(\sum_k | r_{j+k} |^p \right)^{1/p} \leqq C \left(\sum | r_k |^p \right)^{1/p}. \end{aligned}$$

Since, by (3.7),

$$\| Ls \|_p = \left\| \sum_{i=1}^{m+K} c_i \Delta_i^{1/p-1/2} B^*_{i,p} \right\|_p \leqq C \max_i \bar{\Delta}_i^{1/p-1/2} \left\{ \sum_{i=1}^{m+K} | c_i |^p \right\}^{1/p}$$

and furthermore

$$\begin{aligned} \left(\sum_{j=1}^{m+K} | r_j |^p \right)^{1/p} &\leqq C \left\{ \sum_j \left| \int \Delta_j^{-1/2} B_j^* LF \right|^p \right\}^{1/p} \\ &\leqq C \max_j \bar{\Delta}_j^{1/p'-1/2} \left\{ \sum_j \| LF \|^p_{L_p[I_j]} \right\}^{1/p} \\ &\leqq C \max_j \bar{\Delta}_j^{1/2-1/p} \| LF \|_p , \end{aligned}$$

the combination of all these inequalities gives (4.8).

Using Theorems 4.1 and 4.2 we can now establish the desired error bounds for L-spline interplation.

THEOREM 4.3. *Fix $1 \leqq p \leqq \infty$. Then for $\bar{\Delta}$ sufficiently small, the solution $s = s(F)$ of the best interpolation problem* (4.3) *satisfies*

$$\| D^j(F - s)\|_p \leqq C_1 \frac{\bar{\Delta}^m}{\underline{\Delta}^j} \| LF \|_p , \qquad 0 \leqq j \leqq m \tag{4.9}$$

if $F \in L_p^m[a, b]$, and

$$\| D^j(F - s)\|_p \leqq C \frac{\bar{\Delta}^{2m}}{\underline{\Delta}^j} \| L^*LF \|_p \, R_j(\bar{\Delta}/\underline{\Delta}), \qquad 0 \leqq j \leqq 2m - 1 \tag{4.10}$$

if $F \in L_p^{2m}[a, b]$. Here the constant C_1 does not depend on F or Δ while $R_j(\bar{\Delta}/\underline{\Delta}) = (\bar{\Delta}/\underline{\Delta})^{\max(m, 2m-j)+|1/p-1/2|}$.

Proof. By inequalities (3.7) and (3.9) in [11] we have

$$\| D^j(F - s(F))\|_p \leqq C \frac{\bar{\Delta}^m}{\underline{\Delta}^j} \| L(F - s(F))\|_p , \qquad 0 \leqq j \leqq m. \tag{4.11}$$

(Actually these inequalities are stated there only for $p = 2$, but the proof carries over immediately to general p.) Coupling (4.11) with (4.8) yields (4.9).

To prove (4.10), we use the standard trick of working with an intermediate approximation. Let $\mathscr{S}_{L^*L} = \{f \in L_2^{2m-1}[a, b]: f|_{I_i} \in N_{L^*L}, \ i = 0, 1,..., k\}$. By the methods above we can construct a quasi-interpolant $\tilde{Q}$ mapping $L_p[a, b]$ onto $\mathscr{S}_{L^*L}$. Let $g = F - \tilde{Q}F$. Then theorem 4.1 for this quasi-interpolator implies

$$\| D^i g \|_p \leqq C \frac{\bar{\Delta}^{2m}}{\underline{\Delta}^i} \cdot \| L^*LF \|_p , \qquad 0 \leqq i \leqq 2m - 1. \tag{4.12}$$

Since $s(\tilde{Q}F) = \tilde{Q}F$, it follows that

$$\| D^j(F - s)\|_p = \| D^j g - D^j s(g)\|_p \leqq \| D^j g \|_p + \| D^j s(g)\|_p . \tag{4.13}$$

We need to estimate the second term. Let $m \leqq j \leqq 2m - 1$. Then by a Bernstein–Markov-type inequality for L^*L-splines (cf. [12]),

$$\begin{aligned} \| D^j s(g)\| &\leqq C \underline{\Delta}^{m-j} \| D^m s(g)\|_p \\ &\leqq C \underline{\Delta}^{m-j} (\| D^m(s(g) - g \|_p + \| D^m g \|_p). \end{aligned} \tag{4.14}$$

Using (4.8) and (4.11)–(4.12), we obtain

$$\| D^m(s(g) - g)\|_p \leqq C(\bar{\Delta}/\underline{\Delta})^m \| L(s(g) - g)\|_p \leqq C(\bar{\Delta}/\underline{\Delta})^{m+|1/p-1/2|} \| Lg \|_p$$

$$\leqq C(\bar{\Delta}/\underline{\Delta})^{m+|1/p-1/2|} \sum_{i=0}^{m} \| a_i \|_\infty \| D^i g \|_p$$

$$\leqq C(\bar{\Delta}/\underline{\Delta})^{2m+|1/p-1/2|} \bar{\Delta}^m \| L^*LF \|_p$$

Substituting this in (4.14) and then in (4.13), we obtain (4.10) for $m \leqq j \leqq 2m - 1$.

Now let $0 \leqq j \leqq m$. Then (4.10) follows from the fact (cf. [11]) that

$$\| D^j(F - s)\|_p \leqq C\bar{\Delta}^{m-j} \| D^m(F - s)\|_p . \quad \blacksquare$$

5. REMARKS

1. If L is such that N_L has a basis $\{u_i\}_1^m$ which is an ECT-system throughout $[a, b]$, then the LB-splines and the dual basis can be constructed without the assumption that $\bar{\Delta}$ be sufficiently small. In this case the L-splines are called Tchebycheffian splines. Tchebycheffian B-splines were first constructed by Karlin [13]. Here we have followed the construction and normalization of Marsden [15]. For more on the construction of local support bases for generalized spline spaces, see Jerome [8] and Jerome and Schumaker [10].

2. Our approach to constructing a dual basis for the Tchebycheffian B-splines follows that of de Boor [3] used for the polynomial B-splines. There explicit numerical bounds for the norms of the linear functionals could be obtained.

3. The quasi-interpolant constructed in Section 4 is only one of a large collection of possible quasi-interpolants. It is possible that analogs of the quasi-interpolants of Lyche and Schumaker [14] could also be found for L-splines. Their advantage was that the bounds on (lower) derivatives could be established which did not depend on $\underline{\Delta}$. Theorem 4.1 gives bounds on the distance from L-spline spaces to various calsses of smooth functions. For related results of this type, see Jerome [8] and Johnen and Scherer [12]. It is also possible to sharpen Theorem 4.1 to give estimates in terms of certain K-functionals and/or generalized moduli of smoothness—see DeVore [7] for the polynomial spline case.

4. Bounds on the error of L-spline interpolation in terms of $\| LF \|_2$ or $\| L^*LF \|_2$ have been established in several papers; see, e.g., Swartz and Varga [18], Jerome and Varga [11], and Varga [19]. For polynomial splines, de Boor [2] gave estimates in terms of the ∞-norm—(he actually considered the some-

what different and more complicated case of a biinfinite set of interpolation conditions. The results given here can be extended to include more general interpolation conditions (cf. Varga [19]), and can also be made local (cf. [5, 11] in the polynomial case). Biinfinite sets of interpolation conditions as well as periodic problems could also be considered.

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